

A SIMPLE PRESENTATION OF THE HANDLEBODY GROUP OF GENUS 2*

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ABSTRACT. For genus $g = 2$ I simplify Wajnryb's presentation of the handlebody group.

1. INTRODUCTION

Let $S_{g,h,k}$ be a closed surface of genus g , h boundary components and k distinguished points. I'll use the notation S_g for $S_{g,0,0}$. The mapping class group, $\mathcal{M}_{g,h,[k]}$ is the group of all isotopy classes of orientation preserving homeomorphisms which keep the boundary and the distinguished points pointwise fixed. $\mathcal{M}_{g,h,k}$ is the group of all isotopy classes of orientation preserving homeomorphisms which keep the boundary pointwise fixed, and is also fixing the set of distinguished points eventually permuting them. We have $\mathcal{M}_{g,h,[k]} \hookrightarrow \mathcal{M}_{g,h,k}$. I'll use the notation \mathcal{M}_g for $\mathcal{M}_{g,0,0}$. There are some important instances of these groups. For example the braid group B_n is shown in [1], theorem 1.10, to be isomorphic with $\mathcal{M}_{0,1,n}$, and so the pure braid group P_n is isomorphic with $\mathcal{M}_{0,1,[n]}$.

For a long time it was an open problem to obtain a presentation for $\mathcal{M}_{g,1,0}$. In [2] MacCool proved, using purely algebraic methods, that $\mathcal{M}_{g,1,0}$ is finitely presented for any genus g . Hatcher and Thurston in [4] made a crucial breakthrough in the subject developing an algorithm for obtaining a finite presentation for $\mathcal{M}_{g,1,0}$. Using this algorithm, Harer in [5] obtained a finite, explicit presentation. Finally it was Wajnryb, who in [8] gave a simple presentation for $\mathcal{M}_{g,0,0}$ and $\mathcal{M}_{g,1,0}$.

Using similar techniques as in [4], he found in [9] a presentation of the handlebody group. The handlebody group is the subgroup of \mathcal{M}_g formed by all the isotopy classes of orientation preserving homeomorphisms of S_g , which extend to the entire handlebody H_g . I'll denote it by \mathcal{H}_g . The presentation obtained by Wajnryb in [9] is long and complicated.

In this note I will give a simple presentation for \mathcal{H}_2 starting from Wajnryb's presentation. Such a simplification for higher genera doesn't work.

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In Figure 1 it is shown a system of curves on the surface S_g . The isotopy classes of the curves α_i 's will be called meridians, and those of β_i 's will be called longitudes.

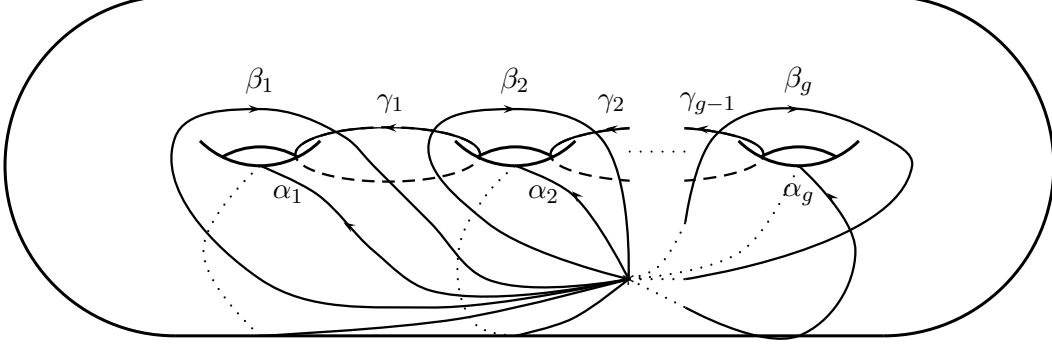


FIGURE 1. Surface of genus g

Definition 1. A (positive) Dehn-twist with respect to a simple closed curve γ , denoted by T_γ , is a homeomorphism of the oriented surface S_g , which is supported in a regular neighborhood of γ and is obtained as follows: one cuts open the surface along γ and rotates one end of it with 360° to the right and then glue back the surface. This is done in such a way that on the boundary and the complement of the regular neighborhood, T_γ is the identity map.

I'll call γ the support of the Dehn-twist.

The effect of a positive T_γ on any segment which intercepts the curve γ transversally in one point is as follows: cut the segment at the interception point and rotate to the right once around γ . The following is a well known result (see [9]):

Lemma 2. *If h is a homeomorphism of the surface S_g , and T_γ is a Dehn-twist, then $T_{h(\gamma)} = hT_\gamma h^{-1}$.*

Remark 3. I'll use the following notation $h*g = hgh^{-1}$. So in this notation Lemma 2 can be written $T_{h(\gamma)} = h * T_\gamma$

Roman letters will be used for a Dehn-twist with a support denoted by the corresponding Greek letter. For the curves in Figure 1 we have $a_i = T_{\alpha_i}$, $b_i = T_{\beta_i}$, $c_i = T_{\gamma_i}$.

Definition 4. A meridinal curve on the surface is one which represents a non-trivial homotopy class in $\pi_1(S_g, *)$ and bounds a properly embedded disc in the handlebody H_g .

Consider N_α the normal subgroup of $\pi_1(S_g, *)$ generated by the homotopy classes of the meridians $\alpha_1, \dots, \alpha_g$ (see Figure 1).

Let $\# : \mathcal{M}_g \longrightarrow \text{Aut}(\pi_1(S_g, *))$ the homomorphism which takes a homeomorphism into the automorphism of the fundamental group of the surface, and the image of a homeomorphism will be denoted with a $\#$ -subscript ($h \mapsto h_\#$). In [3] Griffiths shows that $h \in \mathcal{H}_g$ if and only if $h_\#(N_\alpha) \subset N_\alpha$. A set of generators for \mathcal{H}_g was obtained by Suzuki in [7].

Wajnryb has described in [9] an algorithm for getting a presentation of the handlebody group. This is similar to Hatcher–Thurston’s algorithm in [4] and to Wajnryb’s algorithm in [8]. For the handlebody H_g of genus g there exists an associated 2-dimensional complex X , called the cut-system complex of the handlebody (different from the one constructed for a surface S_g). The vertices are cut-systems (collection of g disjoint meridinal curves). Another cut-system is obtained if we replace one curve with another meridinal curve disjoint from all the curves in the cut system. Such vertices are joined by an edge. Moreover to any triangle we associate a face, thus we obtain a 2-dimensional simplicial complex, called X .

\mathcal{H}_g acts transitively on the vertices of X , and this action can be extended to a simplicial action on X . Using this action Wajnryb obtains the presentation for \mathcal{H}_g .

In Section 2, I will give some details about Wajnryb’s presentation for any g , and in Section 3 the reduction I obtained for the case \mathcal{H}_2 .

2. WAJNRYB’S ALGORITHM

For the convenience of the reader I will describe in this section in some detail Wajnryb’s algorithm. He proved in [9], Theorem 13 that the complex X , decribed above is connected and simply connected. The 0-skeleton $X^{(0)}$ has a preferred vertex represented by the cut-system formed by the collection of the g meridians, one for each handle. It is denoted $\vec{v}_0 = \langle \alpha_1, \alpha_2, \dots, \alpha_g \rangle$. Wajnryb’s algorithm has a few steps.

Step 1: Find a presentation of the stabilizer of \vec{v}_0 denoted by \mathcal{K} . The homeomorphisms in \mathcal{K} either preserve \vec{v}_0 pointwise or permutes the α_i ’s or changes their respective orientations. A presentation for \mathcal{K} can be found using the following exact sequences.

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^g \longrightarrow \pm\Sigma_g \longrightarrow \Sigma_g \longrightarrow 1 \quad (1)$$

$$1 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{K} \longrightarrow \pm\Sigma_g \longrightarrow 1 \quad (2)$$

In both sequences (1) and (2) $\pm\Sigma_g$ is the discrete group of signed permutations. $\pm\Sigma_g$ is the group of permutations of $\{-g, 1-g, \dots, 1, 2, \dots, g\}$ such that $\sigma(-i) = -\sigma(i)$. The homomorphism $\pm\Sigma_g \longrightarrow \Sigma_g$ is the forgetting sign homomorphism and the sequence (1) splits. \mathcal{K}_0 is the subgroup of \mathcal{K} , fixing all the α_i ’s pointwise. To find a presentation of \mathcal{K}_0 , one needs first a presentation for $\mathcal{M}_{0,2g,0}$ which can be obtained from the following diagram.

$$\begin{array}{ccccccc}
& & & 1 & & & \\
& & & \downarrow & & & \\
& & & \mathbb{Z} & & & \\
& & & \downarrow & & & \\
1 & \longrightarrow & \mathbb{Z}^{2g} & \longrightarrow & \mathcal{M}_{0,2g+1,0} & \longrightarrow & \mathcal{M}_{0,1,[2g]} \longrightarrow 1 \quad (3) \\
& & & \downarrow & & & \\
1 & \longrightarrow & \pi_1(S_{0,2g,1}, *) & \longrightarrow & \mathcal{M}_{0,2g,1} & \longrightarrow & \mathcal{M}_{0,2g,0} \longrightarrow 1 \\
& & & \downarrow & & & \\
& & & 1 & & &
\end{array}$$

Using this presentation of $\mathcal{M}_{0,2g,0}$ and the following exact sequence

$$1 \longrightarrow \mathbb{Z}^g \longrightarrow \mathcal{M}_{0,2g,0} \longrightarrow \mathcal{K}_0 \longrightarrow 1 \quad (4)$$

one gets a presentation of \mathcal{K}_0 .

Step 2: Other cut-systems can be obtained using "translations". Such a translation is given by the homeomorphism $r_{i,j}$ which changes α_j into $\gamma_{i,j}$ keeping all the other α_i 's $i \neq j$ fixed (see [9]). Wajnryb, using connectedness of X , proved that the generators of \mathcal{K} together with the elements $r_{i,j}$ generate \mathcal{H}_g .

Step 3: There are finitely many edge orbits modulo the action of \mathcal{H}_g . To each edge we associate its stabilizer $\mathcal{K}_{i,j}$. Relations are coming from the conjugations $r_{i,j}hr_{i,j}^{-1}$ and $r_{i,j}^{-1}hr_{i,j} \in \mathcal{K}$ for any $h \in \mathcal{K}_{i,j}$. For more details see [9].

Using the above described algorithm he was able to find a presentation of the handlebody group which, unfortunately, is very complicated.

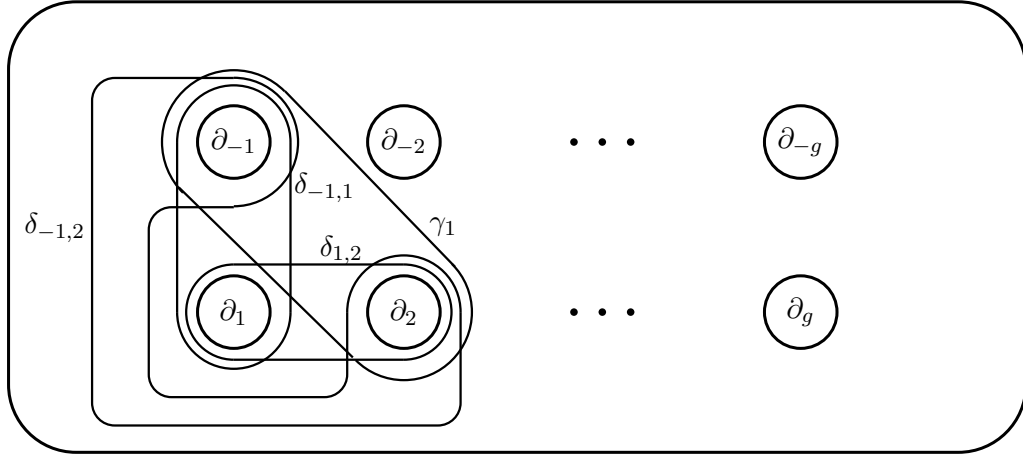


FIGURE 2. A disc with $2g$ holes

Let me give a brief description of the homeomorphisms used in Wajnryb's presentation. In Figure 2 we see some particular meridinal curves. The

curves denoted by γ_1 are the same in both Figure 1 and 2. A positive Dehn-twist is considered to be taken in a counterclockwise direction. Looking carefully at Definition 1 one sees that the Dehn-twist does not depend on the orientation of the curve.

A half Dehn-twist on $\delta_{-1,1}$ is the twist of the first knob and will be denoted by o . It changes the orientation of both the first meridian and longitude. From its definition we see that $o^2 = d_{-1,1}$. It is easily seen that $o(\delta_{1,2}) = \gamma_1$. Using Remark 3 we get that $o * d_{1,2} = c_1$. Other important homeomorphisms on the surface are t_i which exchange the meridians α_i and α_{i+1} , fixing the others. Another homeomorphism which exchanges α_i and α_{i+1} , also exchanging β_i and β_{i+1} , is $k_i = a_i a_{i+1} t_i d_{i,i+1}^{-1}$. I'll mention also z , which is a rotation of the surface about the z-axis (the z-axis is considered to pierce the surface in its center of symmetry as drawn in Figure 1, with positive direction from bellow to above), changing the i -th hole into the $g - i + 1$ -st hole, considering that $i > 0$. From Figure 2 it is clear that a curve $\delta_{i,j}$ is the one which encloses in one of the two regions determined on the disc, the holes ∂_i and ∂_j . The Dehn-twists with these supports can be considered to be the generators of the $P_{2g} \simeq \mathcal{M}_{0,1,[2g]}$. There is one homeomorphism, z_j which belongs to the stabilizer of an edge of type (i, j) in the 1-skeleton of X . In fact z_j has the form $z_j = k_{j-1} k_{j-1} \cdots k_{g+j-1} z$ and it is not the conjugation of z by the product of k_i 's.

In the case $g = 1$ it is proved by Wajnryb in [9], Theorem 14, and detailed in [6], Theorem 2.2 that $\mathcal{H}_1 \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

In the next statement I will use the same notations as in [9], and I will write explicetely Wajnryb's presentation for the case of genus $g = 2$. I will also introduce in the presentation some generators together with their defining relations.

Theorem 5 (Wajnryb). *A presentation of \mathcal{H}_2 is given by:*

Generators: $a_1, a_2, d_{-2,-1}, d_{-21}, d_{-22}, d_{-11}, d_{-12}, d_{12}, o, o_2, t, r, z, e$.

Defining relations:

- (D1) $o^{-1} t^{-1} o^{-1} * d_{12} = d_{-2,-1}$
- (D2) $t^{-1} o^{-1} * d_{12} = d_{-21}$
- (D3) $o^{-1} * d_{12} = d_{-12}$
- (D4) $t^{-1} d_{12} * d_{-11} = d_{-22}$
- (D5) $o_2 = t d_{12}^{-1} * o$
- (D6) $z = a_1^{-1} a_2^{-1} o t o d_{12}$
- (D7) $e = o z o^{-1} z$
- (P1) $a_1 \rightleftharpoons a_2 ; a_i \rightleftharpoons d_{kl}$
- (P2.1) $d_{-2,-1}^{-1} d_{-21} d_{-2,-1} = d_{-21} d_{-11} d_{-21} d_{-11}^{-1} d_{-21}^{-1}$
- (P2.2) $d_{-2,-1}^{-1} d_{-11} d_{-2,-1} = d_{-21} d_{-11} d_{-21}$
- (P2.3) $d_{-2,-1}^{-1} d_{-22} d_{-2,-1} = d_{-22} d_{-12} d_{-22} d_{-12}^{-1} d_{-22}^{-1}$
- (P2.4) $d_{-21}^{-1} d_{-22} d_{-21} = d_{-22} d_{12} d_{-22} d_{12}^{-1} d_{-22}^{-1}$
- (P2.5) $d_{-11}^{-1} d_{-22} d_{-11} = d_{-22}$

- (P2.6) $d_{-2-1}^{-1}d_{-12}d_{-2-1} = d_{-22}d_{-12}d_{-22}^{-1}$
(P2.7) $d_{-21}^{-1}d_{-12}d_{-21} = d_{-22}d_{12}d_{-22}^{-1}d_{12}^{-1}d_{-12}d_{12}d_{-22}d_{12}^{-1}d_{-22}^{-1}$
(P2.8) $d_{-11}^{-1}d_{-12}d_{-11} = d_{-12}d_{12}d_{-12}d_{12}^{-1}d_{-12}^{-1}$
(P2.9) $d_{-2-1}^{-1}d_{12}d_{-2-1} = d_{12}$
(P2.10) $d_{-21}^{-1}d_{12}d_{-21} = d_{-22}d_{12}d_{-22}^{-1}$
(P2.11) $d_{-11}^{-1}d_{12}d_{-11} = d_{-12}d_{12}d_{-12}^{-1}$
(P3) $d_{-2-1}d_{-21}d_{-12}d_{-22}d_{-11}d_{-12}d_{12} = a_1^4a_2^4$
(P4.1) $d_{-11}d_{-12}d_{12} = a_1^2a_2^2$
(P4.2) $d_{-21}d_{-22}d_{12} = a_1^2a_2^2$
(P4.3) $d_{-2-1}d_{-22}d_{-12} = a_1^2a_2^2$
(P4.4) $d_{-2-1}d_{-21}d_{-11} = a_1^2a_2^2$
(P5) *empty for genus 2*
(P6) $o^2 = d_{-11} ; t^2 = d_{12}d_{-2-1}a_1^{-2}a_2^{-2}$
(P7) $t * a_1 = a_2 ; o \rightleftharpoons a_i, \quad i = 1, 2$
(P8) $t \rightleftharpoons d_{12} ; otot = toto ; o \rightleftharpoons d_{-22}$
(P9) $r^2 = a_2^{-4}o_2d_{12}o_2d_{12}^{-1}$
(P10 a) $r * a_2 = d_{12} ; r \rightleftharpoons a_1$
(P10 b – d) *empty for genus 2*
(P10 e) $r \rightleftharpoons e$
(P10 f) $r * d_{12} = a_2$
(P10 g) $r * d_{-2-1} = d_{-11}d_{-12}d_{12}a_1^{-2}a_2^{-1}$
(P10 h – k) *empty for genus 2*
(P11) $rtr = trt$
(P12) *empty for genus 2*

3. A SIMPLE PRESENTATION FOR \mathcal{H}_2

In this section I will prove the following:

Theorem 6. *There is a simple presentation of \mathcal{H}_2 .*

Generators: a_1, a_2, d, o, t, r .

Relations:

$$\begin{aligned}
d \rightleftharpoons oto ; \quad odod &= a_1^2a_2^2 ; \quad o^2 \rightleftharpoons t^{-1}d ; \quad z = a_1^{-1}a_2^{-1}otod ; \\
a_1 \rightleftharpoons a_2 ; a_i \rightleftharpoons d ; \quad t^2 &= d^2a_1^{-2}a_2^{-2} ; \quad o \rightleftharpoons a_i ; t * a_1 = a_2 ; \\
t \rightleftharpoons d ; otot &= toto ; \quad r^2 = d^{-2}a_1^2a_2^{-2} ; \quad r * a_2 = d_{12} ; r \rightleftharpoons a_1 ; \\
r \rightleftharpoons ozo^{-1}z ; \quad rtr &= trt.
\end{aligned}$$

Proof. Use (P4.4) and (P4.1) in (P3) and also the commuting relations (P1) to get

$$(P3)' \quad d_{-11} = d_{-22}$$

Replace d_{-22} with d_{-11} in relations (P4.1)–(P4.4), denoted (P4.1)'–(P4.4)'. Using the commuting relations (P1) and (P4.3)' one gets

$$(P4.1)'' \quad d_{12} = d_{-2-1}.$$

Rewrite again (P4.2)'–(P4.4)', and use (P4.1)'' to obtain (P4.2)''–(P4.4)''. Cojugate (P4.4)'' with d_{12}^{-1} and modulo the commuting relations (P1) one gets again (P4.2)''. So (P4.4)'' is redundant. So far (P3)–(P4.4) look like:

$$\begin{aligned} (P3)' & \quad d_{-11} = d_{-22} \\ (P4.1)'' & \quad d_{-2-1} = d_{12} \\ (P4.2)'' & \quad d_{-21} = a_1^2 a_2^2 d_{12}^{-1} d_{-11}^{-1} \\ (P4.3)'' & \quad d_{-12} = a_1^2 a_2^2 d_{-11}^{-1} d_{12}^{-1} \\ (P4.4)'' & \quad \text{redundant} \end{aligned}$$

Using the above relations in (P2.1)–(P2.11) (the pure braid relations), these all become trivial modulo the commuting relations (P1). This is the main reduction in the presentation, which does not take place for any higher genus. So, for $g = 2$ the pure braid relations are redundant modulo (P1) and consequences of (P3)–(P4.4).

Using Tietze operations, we replace in the remaining relations the expressions for d_{-2-1} , d_{-21} , d_{-22} , d_{-12} and remove these generators together with (P3)''–(P4.4)''. I will remove also generators d_{-11} , o_2 , e together with first part of (P6), (D5) and (D7) replacing first their expressions everywhere else.

At this moment the presentation looks like this:

Generators : $a_1, a_2, d_{12}, o, t, r, z$

Relations :

$$\begin{aligned} (D1) & \quad o^{-1} t^{-1} o^{-1} * d_{12} = d_{12} \\ (D2) & \quad t^{-1} o^{-1} * d_{12} = a_1^2 a_2^2 d_{12}^{-1} o^{-2} \\ (D3) & \quad o^{-1} * d_{12} = a_1^2 a_2^2 o^{-2} d_{12}^{-1} \\ (D4)' & \quad t^{-1} d_{12} * o^2 = o^2 \\ (D6)' & \quad z = a_1^{-1} a_2^{-1} o t o d_{12} \\ (P1) & \quad a_1 \rightleftharpoons a_2 ; a_i \rightleftharpoons d_{12} ; a_i \rightleftharpoons o^2 \\ (P6)' & \quad t^2 = d_{12}^2 a_1^{-2} a_2^{-2} \\ (P7) & \quad t * a_1 = a_2 ; o \rightleftharpoons a_i \\ (P8) & \quad t \rightleftharpoons d_{12} ; o t o t = t o t o ; o \rightleftharpoons o^2 \\ (P9)' & \quad r^2 = a_2^{-4} (t d_{12}^{-1} * o) d_{12} (t d_{12}^{-1} * o) d_{12}^{-1} \\ (P10a) & \quad r * a_2 = d_{12} ; r \rightleftharpoons a_1 \\ (P10e)' & \quad r \rightleftharpoons o z o^{-1} z \\ (P10f)' & \quad r * d_{12} = a_2 \\ (P10g)' & \quad r * d_{12} = o^2 a_1^2 a_2^2 o^{-2} d_{12}^{-1} d_{12} a_1^{-2} a_2^{-1} \\ (P11) & \quad r t r = t r t \end{aligned}$$

Let $d_{12} = d$ (it is the only d_{ij} left). In the above (D1) is equivalent with (D1)' $o t o \rightleftharpoons d$.

Rewrite (D2) and use (D1)' to get (D2)':

$$t^{-1} o^{-1} d o t = a_1^2 a_2^2 d^{-1} o^{-2} \Leftrightarrow t^{-1} o^{-1} d o t o = a_1^2 a_2^2 d^{-1} o^{-1} \Leftrightarrow$$

$$(using\ P1) \Leftrightarrow t^{-1}o^{-1}otod = a_1^2a_2^2d^{-1}o^{-1} \Leftrightarrow od = a_1^2a_2^2d^{-1}o^{-1} \Leftrightarrow \\ (D2)' \quad odod = a_1^2a_2^2.$$

Conjugating the above with d and using (P1) we get a redundant relation:

$$(D2)'' \quad dodo = a_1^2a_2^2 = odod.$$

Rewrite (D3) using (D2)'' as follows:

$$o^{-1}do = a_1^2a_2^2o^{-2}d^{-1} \Leftrightarrow o^{-1}dodo = a_1^2a_2^2o^{-1}(using\ (D2)'') \Leftrightarrow \\ o^{-1}a_1^2a_2^2 = a_1^2a_2^2o^{-1}(trivial\ because\ of\ (P7)).$$

Relation(D4)' is equivalent with $t^{-1}d_{12} \rightleftharpoons o^2$.

In (P1) the last part, $o^2 \rightleftharpoons a_i$, is redundant from the second part of (P7) which is $o \rightleftharpoons a_i$.

In (P8) we can get rid of $o \rightleftharpoons o^2$.

Lastly in (P10g)', because of the commuting relations (P1) and (P7) we'll get (P10f)'. So (P10g)' is redundant.

Rewrite (P9)'. Use for this (P8), (D2)'', (P1) to get $r^2 = d^{-2}a_1^2a_2^{-2}$.

Because (P10f)' and the above one easily gets that the first part of (P10a) is redundant.

This is the presentation in the statement of Theorem 6

□

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